Descriptive Set Theory
Lecture

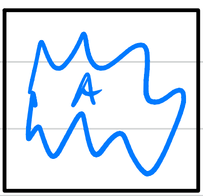
(d) For the Cantor space $2^{\mathbb{N}}$, we can define the power of any measure on $2 := \{0,1\}$ called a coin-flip or Bernoulli measure. Formally, let ν be a prob. meas. on 2, so $\nu = p \cdot \delta_1 + (1-p) \cdot \delta_0$ and let $\mu := \nu^{\mathbb{N}}$. This $\nu^{\mathbb{N}}$ is defined by defining it on basic clopen sets and extending by Caratheodory.
$\nu^{\mathbb{N}}([01101]) := \nu(0) \cdot \nu(1) \cdot \nu(1) \cdot \nu(0) \cdot \nu(1)$.
To apply the extension theorem, one needs to check that this definition is ctbly-additive on basis clopen sets.
When $p = \frac{1}{2}$, we call this the fair coin-flip measure.
Thinking of $2^{\mathbb{N}}$ as the abelian group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ with coordinate-wise addition (equivalently, the group $(\mathcal{P}(\mathbb{N}), \Delta)$), the fair coin-flip measure is invariant under translation.

(e) For any locally compact top gp (e.g. $\mathbb{R}^d$, $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$, S^1), Haar proved that there exists a nonzero Borel measure μ that is left-translation invariant, finite on compact sets, and regular (i.e. the measure of every measurable set can be approximated from above by open and from below by closed sets). Such a measure is unique up to constant multiplication and is called a Haar measure. When the group is compact, we normalize the measure to be probability. All the examples above, except (c), are Haar measures.

(f) Counting measure on any set X is just $\mu(A) := \begin{cases} \infty & \text{if } |A| = \infty \\ |A| & \text{if } |A| < \infty \end{cases}$ i.e. $\mu(\{x\}) = 1 \ \forall x \in X$. In analysis and DST, we like to only work with σ-finite measures, so the counting measure is most useful when X is ctbl.

Null and measurable sets. Given a Borel measure μ on a top space X, we call a subset $A \subseteq X$ μ-null if $A \subseteq B$ for some Borel $B \subseteq X$ with $\mu(B) = 0$. Thus, $\mathcal{P}(B)$ consists of null sets.

B



Remark. There are only continuum many Borel subsets of a Polish space, while for any non-atomic on an unctbl Polish space, $\exists\, 2^{\text{contin.}}$ many null sets. E.g. the whole $\mathcal{P}(C)$, where $C \subseteq [0,1]$ is the usual Cantor set, consists of null sets.

Let $\mathrm{NULL}_\mu(X)$ denote the collection of null sets and note that it is a σ-ideal (i.e. closed under subsets and ctbl unions).

There is a strong analogy between $\mathrm{NULL}_\mu(X)$ and $\mathrm{MGR}(X)$, but not everything works interchangeably.

Remark. By exhausting the measure on $\mathbb{R}$ using Cantor sets of positive measure, one builds a meager F_σ subset of $\mathbb{R}$ that's conull, i.e. the complement is null. Thus, category and measure live on disjoint sets.

Def. For a Borel measure μ on a top. space X, a set $A \subseteq X$ is called μ-measurable if $A =_\mu B$ for some Borel set $B \subseteq X$, where $=_\mu$ means $A \Delta B$ is μ-null.

As remarked above, there are way more measurable sets than there are Borel sets. μ is naturally extended to the σ-algebra $\mathrm{MEAS}_\mu(X)$ of all μ-measurable sets and we call this extension the completion of μ and it's usually denoted by $\bar{\mu}$, but we just use μ.

Def. A subset A of a Polish space X is called *universally measurable* if it is measurable wrt any σ-finite (equivalently, probability) Borel measure on X.
why?

These sets also form a σ-algebra.

Call a function $f: X \to Y$, X, Y Polish, *universally measurable* if $f^{-1}(B)$ is universally measurable $\forall B \in \mathcal{B}(Y)$.

Prop. Composition of universally measurable functions is universally measurable (unlike just μ-measurable functions).

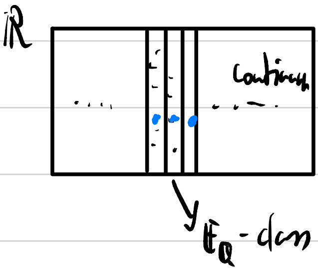
Proof. HW.

We'll show that analytic (= continuous images of Borel sets) are universally measurable.

Measure isomorphism theorem. Let (X, μ) be a Polish space equipped with a nonatomic Borel prob. measure. Then $\exists$ Borel isomorphism $f: X \to [0,1)$ s.t. $f_* \mu = \lambda$, i.e. every nonatomic Borel prob measure on a Polish space is isomorphic to $[0,1)$ with Lebesgue measure.

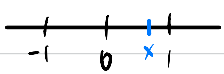
This is a consequence of the Borel isomorphism theorem, which we'll prove later and deduce the measure isomorphism.

Nonmeasurable sets. By the meas. isom. thm, I'll only bother with Lebesgue measure. Let $\mathbb{E}_\mathbb{Q}$ be the orbit equiv. relation of the translation action of $\mathbb{Q}$ on $\mathbb{R}$, i.e. $x \, \mathbb{E}_\mathbb{Q} \, y :\Leftrightarrow x - y \in \mathbb{Q}$. This is known as the *Vitali equivalence relation*. Call a set $A \subseteq \mathbb{R}$ a *transversal* for $\mathbb{E}_\mathbb{Q}$ if A meets every $\mathbb{E}_\mathbb{Q}$-class at exactly 1 point.



Prop. Any transversal A of $\mathbb{E}_\mathbb{Q}$ is λ-non-measurable.

Proof. Suppose A is λ-measurable. Then let $A' := A \cap [-1,1]$.



$$[-1,1] \subseteq \bigcup_{\substack{q \in \mathbb{Q} \\ \cap [-1,1]}} (q + A') \subseteq [-2,2].$$

Thus, this union is nonnull & has finite measure.
On the other hand, this is a disjoint union because A' is a transversal (hence $(q_0+A')\cap(q_1+A')=\emptyset$), and λ is translation invariant, so $\lambda(q+A')=\lambda(A')$.
Thus, $\lambda\left(\bigcup_{q\in\mathbb{Q}\cap[-1,1]} q+A'\right)=\sum_{q\in\mathbb{Q}\cap[-1,1]}\lambda(A')=\infty$ (it can't be null), a contradiction. $\square$

Lebesgue density.

95% lemma for Lebesgue. For every nonnull measurable set $B\subseteq\mathbb{R}^d$, there is an open rectangle $R:=I_1\times\ldots\times I_d$ that is 99% B, i.e. $\frac{\lambda(R\cap B)}{\lambda(R)}\geq 0.99$. (Of course, this is true for any $1-\varepsilon$.)

Proof. For $d=1$. Let $U\supseteq B$ be an open set that is 99% B. Then $U=\bigsqcup_n I_n$ disjoint union of open intervals. Thus one of the I_n has to be 99% B.
For $d>1$, we'll remove the boundaries of rational rectangles and get that any open U is a disjoint union of open rectangles plus a null set. The rest of the argument is the same. $\square$